Int. J. Solids Structures, Vol. 18, No. 1, pp. 13–22, 1982
Printed in Great Britain.

0020-7683/82/010013-10$03.00/0

EXPLICIT OPTIMUM DESIGN

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(*Received* 23 *April* 1980; *in revised form* 9 *January* 1981)

Abstract—The optimization of a trussed type structure of given geometry and material properties can be formulated as an exact and explicit mathematical programming problem in a mixed space of design variables and behaviour variables. Three techniques are presented, corresponding to the three classical analysis methods of structural theory. In the case of a single loading condition without variables linking, the proposed method is very efficient since it eliminates the problem of multiple reanalysis without increasing the dimensionality of the problem and the number of constraints. In the other cases the numerical efficiency of the technique depends on the specific problem to be solved.

1. INTRODUCTION

Since the earliest applications of mathematical programming methods to structural design[1], the approach was plagued by the numerical effort required for the computation of the behavioural functions, such as stresses and displacements, at the successive candidate design points. These quantities are usually implicit functions of the design variables and their evaluation requires, in principle, a reanalysis of the structure for every move in the design space. In fact, much of the research in optimum structural design during the last decade was channelled towards developing methods which necessitate less reanalysis, the implied ultimate goal being the elimination of analysis altogether. In the design of a minimum volume truss of given geometry and elastic material properties, one encounters basically the same problem. The nodal displacements **u** and member stresses are implicit functions of the cross-sections of the elements **a** and their evaluation requires an analysis of the structure for every change in the design vector **a**.

Many methods have been developed for this basic minimization problem, among others the two following families of solution techniques. A first approach substitutes for the implicit behavioural functions approximate explicit expressions, usually truncated Taylor series expansions. Quadratic[2] and linear [3] approximations have been used and the latter seem to be very reliable when developed in terms of the compliances of the structural members[4, 5]. The behavioural functions can also be approximated by single term polynomials and the minimization is then performed using geometric[6] or linear [7] programming. In all cases, exact extensive analyses are periodically required to update the model such as to ensure convergence to the optimum solution.

An alternative approach circumvents the analysis obstacle by appending the analysis variables to the design vector[8]. Consider, for instance, the equilibrium equations of the displacement method

$$\mathbf{K}\mathbf{u} = \mathbf{p} \tag{1}$$

where **K** is the reduced stiffness matrix and **p** is the vector of external nodal loads. In the design space of variables **a** and **u**, the original minimization problem in addition to the equality constraints (1) is a mathematical programming problem with explicit constraints since the stiffness matrix can be expressed as

$$\mathbf{k} = \sum_{i=1}^{m} a_i \mathbf{K}_i \tag{2}$$

where m is the number of elements of the structure, and matrices $\mathbf{K}_i$ are function of element geometry, connectivity and material properties only. Similar considerations lead to explicit constraints in the design space of **a** and $\mathbf{t}_1$, where $\mathbf{t}_1$ is the vector of redundant axial forces [9]. In this case, the basic problem is supplemented with equality constraints which impose geometric compatibility.

As opposed to the approximate model methods, the second family of techniques merits more the denomination of structural synthesis, since design and analysis variables and constraints form one mathematical unit. These methods, however, have serious computational drawbacks. The inclusion of analysis variables and constraints in the mathematical programming formulation invariably expands the design space and increases substantially the number of constraints. This is even more so in the case of multiple loading conditions where a separate set of analysis variables and constraints is required for every loading case.

The purpose of this paper is to show that a judicious choice of design and analysis variables can substantially improve the approach. In the case of a single loading condition, explicit and exact mathematical programming formulations can be constructed in design spaces equal in size to the original space, with no additional constraints. Whereas analysis is introduced in the classical methods through equality constraints, there are no separate analysis equations in the present method. For multiple loading conditions, the design spaces are expanded due to the duplication of the analysis variables and additional constraints are required. In the latter case the method is still very attractive, but its performance when compared to existing techniques is problem dependent.

2. RATIONALE OF THE METHOD

Consider a statically redundant truss subjected to a single load vector $\mathbf{p}$ and let $\mathbf{a}_0$ and $\mathbf{a}_1$ be respectively the cross-sectional areas of a basic statically determinate structure and the redundant bars, and $\mathbf{t}_0$ and $\mathbf{t}_1$ be the corresponding stress resultants. The geometric compatibility equations can be expressed as

$$(\mathbf{F}_1 - \mathbf{N}_1)\mathbf{t}_1 = \mathbf{e}_1^{P} \tag{3}$$

where $\mathbf{F}_1$ is the diagonal matrix of the redundant member flexibilities, $\mathbf{e}_1^P$ are the relative displacements of the end nodes of the redundant members due to the external loads, $\mathbf{N}_1$ is a matrix of influence coefficients N_{ij}, and N_{ij} is the relative displacement of the nodes of redundant member i due to the self-equilibrating unit forces at the nodes of redundant member j. Equations (3) are then solved for the redundant forces $\mathbf{t}_1$. However, since

$$\mathbf{F}_1\mathbf{t}_1 = \mathbf{T}_1\mathbf{f}_1 \tag{4}$$

where $\mathbf{T}_1$ is the diagonal matrix corresponding to $\mathbf{t}_1$ and $\mathbf{f}_1$ is the vector of diagonal elements of $\mathbf{F}_1$, eqn (3) can be rewritten as

$$\mathbf{f}_1 = \mathbf{T}_1^{-1}(\mathbf{e}_1^P + \mathbf{N}_1\mathbf{t}_1). \tag{5}$$

This is an alternative form of the geometric compatibility requirements. As a result, compatibility can be formulated two-fold. Given the areas of the basic and redundant members $\mathbf{a}_0$ and $\mathbf{a}_1$ and external loads $\mathbf{p}$, the redundant forces are obtained from the solution of eqn (3). Alternatively, given the areas of the basic members $\mathbf{a}_0$, a set of redundant forces $\mathbf{t}_1$ and the external loads $\mathbf{p}$, one can obtain the areas of the redundant bars $\mathbf{a}_1$ by evaluating the r.h.s. of eqn (5). Obviously, a structure composed of these areas and subject to $\mathbf{p}$ would produce the same loads $\mathbf{t}_1$ in its redundant bars. The compatibility equations are usually not presented as in eqns (5), since they are of no practical use for the structural analyst. For the structural designer, on the other hand, eqns (5) are most interesting. They indicate that in terms of variables $\mathbf{a}_0$ and $\mathbf{t}_1$, the entire design problem is explicit. For example, the ith equation of system (5) is

$$f_i = \left(e_i^P + \sum_{j=1}^{r} N_{ij}t_j\right)\Big/ t_i \tag{6}$$

where r is the number of redundants (subscripts 1 have been omitted for the sake of clarity). It is instructive to note that e_i^P and N_{ij} are explicit functions of the basic cross-sections $\mathbf{a}_0$ and independent of $\mathbf{a}_1$. The stress resultants in the basic structure $\mathbf{t}_0$ are explicit functions of $\mathbf{t}_1$ and so are $\mathbf{u}$ and $\mathbf{e}$. In addition, the redundant areas are explicit functions of $\mathbf{a}_0$ and $\mathbf{t}_1$ (6) and therefore also the volume.

These equations were developed based on a force method of structural analysis. Similar equations can be constructed when considering a displacement type of analysis or a force-displacement method of analysis (Reissner). In all three cases, an exact explicit mathematical programming problem can be formulated in a suitable design space as will be shown in the next section.

3. EXPLICIT ANALYSIS

Consider a statically redundant truss of given geometry and material properties with m axial members and n displacement degrees of freedom ($m \geq n$). The elements of the structure obey Hooke's law and their constitutive law is given by the axial stiffnesses $\mathbf{s}$ or flexibilities $\mathbf{f}$, also referred to symbolically by $\mathbf{h}$. The behaviour of the structure is defined by the nodal displacements $\mathbf{u}$, element elongations $\mathbf{e}$ and element end loads $\mathbf{t}$. These variables are related by the equations governing the structural response:

$$\text{Equilibrium equations: } \mathbf{Q}_0\mathbf{t}_0 + \mathbf{Q}_1\mathbf{t}_1 = \mathbf{p} \tag{7a}$$

$$\text{Hooke's law: } \mathbf{S}_0\mathbf{e}_0 = \mathbf{t}_0; \quad \mathbf{S}_1\mathbf{e}_1 = \mathbf{t}_1 \tag{7b}$$

$$\text{Elongation-displacement relations: } \mathbf{R}_0\mathbf{u} = \mathbf{e}_0; \quad \mathbf{R}_1\mathbf{u} = \mathbf{e}_1 \tag{7c}$$

where $\mathbf{Q}_i = \mathbf{R}_i^T (i = 0, 1)$ are functions of node coordinates and member connectivity only and $\mathbf{S}_i (i = 0, 1)$ are diagonal matrices of element stiffnesses. Subscripts "0" relate to a basic statically determinate substructure and subscripts "1" correspond to the remaining redundant bars.

Hooke's law can also be expressed in terms of the flexibilities

$$\mathbf{F}_0\mathbf{t}_0 = \mathbf{e}_0; \quad \mathbf{F}_1\mathbf{t}_1 = \mathbf{e}_1 \tag{8}$$

where $\mathbf{F}_i = \mathbf{S}_i^{-1}$ $(i = 0, 1)$.

In classical analysis the structural properties $\mathbf{h}$ are given quantities and the $(2m + n)$ field variables $\mathbf{u}$, $\mathbf{e}$ and $\mathbf{t}$ (behaviour variables in structural optimization terminology) are obtained from the $(2m + n)$ eqns (7). However, one could conceive a more general structural analysis problem by including the m constitutive laws $\mathbf{h}$ in the variables set. This generates a problem with $(3m + n)$ unknowns and $(2m + n)$ equations. The solution of these equations would therefore be a function of m predetermined parameters. In the usual approach the parameters are the constitutive law of the structure $\mathbf{h}$. This leads invariably to the solution of a system of linear equations in a subset of behaviour variables. The coefficients matrix of these equations is a function of the parameters and any change in $\mathbf{h}$ requires, in principle, a new solution of the equations. We will call this approach implicit analysis since the variables of the problem are implicit functions of the parameters.

It will be shown in the following that for each of the three fundamental methods of (implicit) structural analysis, that is, the Force, Displacement and Hybrid (Reissner's) Methods, there exists a corresponding explicit analysis model. In all three cases, the variables are explicit functions of the parameters and for any change in the parameters, the new values of the variables are obtained by evaluating simple algebraic expressions.

(a) *Force method*

The classical approach of the Force Method is: Given the constitutive law of the truss $\mathbf{h}_0$ and $\mathbf{h}_1$, what are the conditions that the redundant forces $\mathbf{t}_1$ have to satisfy in order to maintain geometric compatibility. The compatibility equations can be obtained as follows. A set of compatible elongations $\mathbf{e}$ have to verify (7c)

$$\mathbf{e}_1 = \mathbf{G}\mathbf{e}_0 \tag{9}$$

where

$$\mathbf{G} = \mathbf{R}_1\mathbf{R}_0^{-1}. \tag{10}$$

Rewriting eqn (9) in terms of the axial forces gives with (8)

$$\mathbf{f}_1\mathbf{t}_1 = \mathbf{G}\mathbf{F}_0\mathbf{t}_0. \tag{11}$$

From equilibrium (7a) we have

$$\mathbf{t}_0 = \mathbf{t}_b - \mathbf{G}^T\mathbf{t}_1 \tag{12}$$

$$\text{with } \mathbf{t}_b = \mathbf{Q}_0^{-1}\mathbf{p}. \tag{13}$$

Substitution of eqn (12) in eqn (11) yields the compatibility relations

$$(\mathbf{G}\mathbf{F}_0\mathbf{G}^T + \mathbf{F}_1)\mathbf{t}_1 = \mathbf{G}\mathbf{F}_0\mathbf{t}_b. \tag{14}$$

These equations are identical to eqns (3) previously derived. The variables of the problem are the redundant forces $\mathbf{t}_1$ and the parameters are the flexibilities $\mathbf{f}$. Every change in $\mathbf{f}$ alters the coefficients matrix and requires a new solution of eqns (14) in order to compute $\mathbf{t}_1$.

In the Explicit Force Method the approach is: Given the constitutive law of a basic statically determinate structure $\mathbf{h}_0$ and the forces in the redundant members $\mathbf{t}_1$, find the constitutive law of the redundant components $\mathbf{h}_1$ in order to maintain geometric compatibility. The derivation of the compatibility conditions is essentially similar to what was done previously, except for eqn (11) which is rewritten such as

$$\mathbf{T}_1\mathbf{f}_1 = \mathbf{G}\mathbf{F}_0\mathbf{t}_0. \tag{15}$$

This leads to the compatibility equations

$$\mathbf{f}_1 = \mathbf{T}_1^{-1}\mathbf{G}\mathbf{F}_0(\mathbf{t}_b - \mathbf{G}^T\mathbf{t}_1). \tag{16}$$

As indicated, the variables of the problem, $\mathbf{f}_1$, are explicit functions of the parameters $\mathbf{f}_0$ and $\mathbf{t}_1$.

(b) *Displacement method*

The equations of the Displacement Method can be obtained as follows: Given the constitutive law of the structure $\mathbf{h}_0$ and $\mathbf{h}_1$ find the nodal displacements $\mathbf{u}$ that will maintain nodal equilibrium. Rewriting the equilibrium equations (7a) in terms of elongations using Hooke's law (7b) yields

$$\mathbf{Q}_0\mathbf{S}_0\mathbf{e}_0 + \mathbf{Q}_1\mathbf{S}_1\mathbf{e}_1 = \mathbf{p} \tag{17}$$

and expressing the elongations as a function of the displacements (7c) gives the conditions for nodal equilibrium

$$(\mathbf{Q}_0\mathbf{S}_0\mathbf{R}_0 + \mathbf{Q}_1\mathbf{S}_1\mathbf{R}_1)\mathbf{u} = \mathbf{p} \tag{18}$$

where the term in parenthesis is the reduced stiffness matrix $\mathbf{K}$ of the structure. The coefficients matrix of these equations is a function of the parameters s and any change in these parameters necessitates a new solution of these equations in order to determine the variables $\mathbf{u}$.

Alternatively, in the Explicit Displacement Method the following problem is posed: Given a set of nodal displacements $\mathbf{u}$ and the constitutive law of the redundant bars $\mathbf{h}_1$ find the constitutive law of the basic structure $\mathbf{h}_0$ that will maintain nodal equilibrium. The solution to this problem is very similar to the previous derivation. Equation (17) is rewritten as

$$\mathbf{Q}_0\mathbf{E}_0\mathbf{s} + \mathbf{Q}_1\mathbf{S}_1\mathbf{e}_1 = \mathbf{p} \tag{19}$$

where $\mathbf{E}_0$ is the diagonal matrix corresponding to $\mathbf{e}_0$. Expressing $\mathbf{e}_1$ in terms of $\mathbf{u}$ using eqn (7c), yields the explicit equilibrium conditions

$$\mathbf{s}_0 = \mathbf{E}_0^{-1}(\mathbf{t}_b - \mathbf{G}^T\mathbf{S}_1\mathbf{R}_1\mathbf{u}) \tag{20}$$

where the variables $\mathbf{s}_0$ are explicit functions of the parameters $\mathbf{u}$ and $\mathbf{s}_1$.

(c) *Hybrid method*

The Hybrid analysis equations, which can be derived from Reissner's Functional, express the conditions that nodal displacements $\mathbf{u}$ and redundant forces $\mathbf{t}_1$ have to satisfy if the displacements and force fields are related by a given constitutive law $\mathbf{h}$. Expressing the constitutive law (7b), (8) in terms of $\mathbf{u}$ and $\mathbf{t}_1$ using eqns (7c) and (12) yields the equations

$$\begin{bmatrix} \mathbf{Q}_0\mathbf{S}_0\mathbf{R}_0 & \mathbf{Q}_1 \\ \mathbf{R}_1 & -\mathbf{F}_1 \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{t}_1 \end{Bmatrix} = \begin{Bmatrix} \mathbf{p} \\ \mathbf{o} \end{Bmatrix} \tag{21}$$

where $\mathbf{o}$ is a zero vector of the order of the statical redundancy of the structure. The variables of the problem are $\mathbf{u}$ and $\mathbf{t}_1$ and as previously, the coefficients matrix is a function of the parameters $\mathbf{h}$.

The Explicit Hybrid Method finds the constitutive law of a structure which has given nodal displacements $\mathbf{u}$ and redundant forces $\mathbf{t}_1$. Rewriting the constitutive equations (7b) as

$$\mathbf{s}_i = \mathbf{E}_i^{-1}\mathbf{t}_i \quad (i = 0, 1) \tag{22}$$

where $\mathbf{E}_i$ is the diagonal matrix corresponding to $\mathbf{e}_i$ yields with eqn (12) the required constitutive law

$$\mathbf{s}_0 = \mathbf{E}_0^{-1}(\mathbf{t}_b - \mathbf{G}^T\mathbf{t}_1)$$
$$\mathbf{s}_1 = \mathbf{E}_1^{-1}t_1 \tag{23}$$

where the elongations are obtained from the displacements using eqn (7c). The variables of the problem $\mathbf{h}$ are again explicit functions of the parameters $\mathbf{u}$ and $\mathbf{t}_1$.

The three fundamental analysis methods are summarized symbolically in Table 1, for both the implicit and explicit approach.

4. EXPLICIT DESIGN

We have shown in the previous section that the analysis of a structure of given geometry and material properties could be conceived in a broader sense by considering the structural properties of the members as variables of the problem in addition to the behaviour variables $\mathbf{t}_1$ and $\mathbf{u}$. In this expanded set of variables ($\mathbf{h}_0$, $\mathbf{h}_1$, $\mathbf{u}$, $\mathbf{t}_1$) two vectors are parameters and the remaining two vectors are obtained by imposing that the structure satisfies equilibrium, compatibility and Hooke's law. In classic analysis the parameters are always the structural properties $\mathbf{h}_0$, $\mathbf{h}_1$ and the variables are behaviour variables. These are implicit functions of ($\mathbf{h}_0, \mathbf{h}_1$) and are obtained by solving a system of linear equations. In the present approach the independent variables are suitable combinations of structural and/or behavioural variables. As a result, the variables are explicit in terms of the parameters.

The usual approach in structural optimization is to pose the mathematical programming problem in the space of the structural variables $\mathbf{h}$. The behavioural variables are therefore implicit functions of the design variables. However, if we formulate the minimization problem in the space of ($\mathbf{h}_0, \mathbf{t}_1$), ($\mathbf{u}, \mathbf{h}_1$) or ($\mathbf{u}, \mathbf{t}_1$) the mathematical programming problem is explicit since

Table 1. Implicit vs explicit analysis

	Implicit	equation	Explicit	equation
Force	$[h_0, h_1]\{t_1\} = \{q\}$	14	$h_1 = h_1(h_0, t_1)$	16
Displacement	$[h_0, h_1]\{u\} = \{p\}$	18	$h_0 = h_0(u, h_1)$	20
Hybrid	$[h_0, h_1]\begin{Bmatrix} u \\ t_1 \end{Bmatrix} = \begin{Bmatrix} p \\ 0 \end{Bmatrix}$	21	$h_0 = h_0(u, t_1)$ $h_1 = h_1(u, t_1)$	23

all the behaviour variables and the objective function are explicit functions of the design vector. One will note that in all three cases the size of the design space is equal to the size of the $(\mathbf{h}_0, \mathbf{h}_1)$ design space since the number of nodal displacements $\mathbf{u}$ is equal to the number of bars in the basic structure $\mathbf{h}_0$. The number of constraints is equal in both approaches. For the case of a single loading condition, the proposed method is therefore very interesting, since it bypasses the obstacle of multiple reanalysis without adding constraints or expanding the design space.

The general numerical implementation of the technique calls, however, for some precautions. The mixing of design variables of very different nature can distort the design space and some variable scaling is recommended. Since analysis variables are included in the design vector, the choice of an appropriate initial design is sometimes non-trivial. This is especially the case in the displacement and hybrid approach where the nodal displacements are part of the design vector. A single classical analysis solves usually the problem. To this effect, it is important to realize that non-negativity constraints on the member cross-sections have to be included in the formulation. For instance, a hybrid explicit analysis could very well yield negative stiffnesses for some assumed values of the design vector $(\mathbf{u}, \mathbf{t}_1)$.

Variables linking is usually considered beneficial in an optimization problem since it reduces the dimensionality of the design space[4]. With the present approach the statement is not always true. If the linking is between members of the basic structure or between redundant members it reduces the size of the problem. The explicit force approach, design space $(\mathbf{h}_0, \mathbf{t}_1)$, should be used in the first case and the explicit displacement approach, design space $(\mathbf{u}, \mathbf{h}_1)$, should be used in the latter. If the linking expresses a physical symmetry of the problem, that is, a symmetric structure subjected to a symmetric loading, any of the three methods can be used.

In practice, however, more general cases of variables linking do arise and as a result the explicit optimization problem must be supplemented with equality constraints of the type

$$\mathbf{V}\mathbf{h} = \mathbf{o} \tag{24}$$

where $\mathbf{V}$ is usually a binary matrix.

Multiple loading conditions can be treated similarly but they cause both an expansion of the design space and an addition of equality constraints. In effect for every loading case a separate set of behaviour variables is created. If we consider a force approach the design vector will be the concatenation of $\mathbf{f}_0$ and $\mathbf{t}_1^{(q)}(q = 1, 2, \ldots, c)$ where q is the loading case index and c is the number of loading conditions. Since every assumed internal force vector $\mathbf{t}_1^{(q)}$ can produce a different redundant flexibilities vector (16) the design must be supplemented with equality constraints of the type

$$\mathbf{f}_1^{(q+1)} - \mathbf{f}_1^{(q)} = \mathbf{o} \qquad q = 1, 2, \ldots, c-1. \tag{25}$$

The objective function can be expressed in terms of any $\mathbf{f}_1^{(q)}$ or as a function of the arithmetic mean of the $\mathbf{f}_1^{(q)}$'s.

The two extreme cases for assessing the efficiency of the explicit optimization problem are thus, on the one hand, the design of a truss for a single loading condition without variables linking and on the other hand, the case of multiple loading conditions with variables linking. In the former case the present method is very powerful and probably superior to any other available optimization scheme. The method becomes more involved in the latter case, due to the addition of equality constraints and expansion of the design space and its performance depends strongly on the specific problem at hand.

5. DESIGN SPACES

It is instructuve to compare the design spaces of a same problem using an implicit mathematical programming formulation and an explicit method. The three-bar symmetric truss problem[1] appears to be very suitable for this purpose since the structure is statically redundant and the design space is two-dimensional. The purpose of the optimization scheme is to minimize the volume of a truss of given geometry and material properties submitted to a single loading condition (Fig. 1). The design variables are the cross-sections of the external bars a_1 and the cross section of the middle bar a_2. The axial stresses of the members are constrained by upper and lower bounds, $\bar{\bar{\sigma}}$ and $\bar{\sigma}$, respectively.

In the classical formulation, the optimization variables are the design variables and the mathematical programming equations are:

$$\text{minimize} \quad v = 1(2a_1/c + a_2) \tag{26a}$$

subject to stress contraints

$$\sigma_1(a_1, a_2) \le \bar{\bar{\sigma}} \tag{26b}$$

$$\sigma_2(a_1, a_2) \le \bar{\bar{\sigma}} \tag{26c}$$

$$\sigma_3(a_1, a_2) \ge \bar{\sigma} \tag{26d}$$

and to side constraints

$$a_1 \ge 0 \tag{26e}$$

$$a_2 \ge 0 \tag{26f}$$

where v is the volume of the structure, $1 = 10$ in. is the length of the middle bar, $c = \cos(\pi/4)$, $p = 20{,}000$ lbs $\bar{\bar{\sigma}} = 20{,}000$ psi, $\bar{\sigma} = -15{,}000$ psi and $\sigma_i (i = 1, 2, 3)$ are the axial stresses in the 3 elements of the structure. The design space for this problem is shown in Fig. 2(a). The stress constraints are given by full lines and three typical isovolume lines are shown by dotted lines. The design is constrained by the upper stress limit for bar 1 and the optimum design variables are $a_1 = 0.79$, $a_2 = 0.41$ yielding a minimum volume $v = 26.4$.

Using an explicit force type approach we chose as basic structure the two external bars and the optimization variables are a_1 and the axial force in the redundant element t_2, (all forces normalized with respect to p). From the geometry of the structure, we have (10), (13)

$$\mathbf{G} = \{1 \quad 1\}/2c$$

and

$$\mathbf{t}_b = \{1 \quad 0\}^T.$$

The compatibility equations in terms of a_1 and t_2 become (16)

$$a_2 = a_1 c t_2/(c - t_2). \tag{27}$$

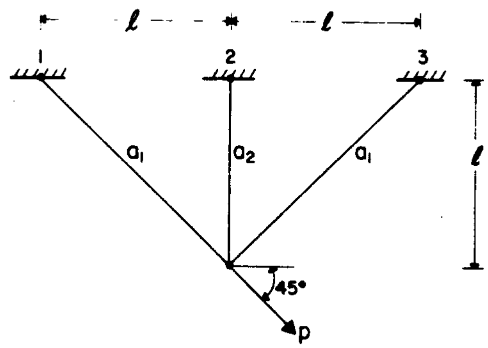
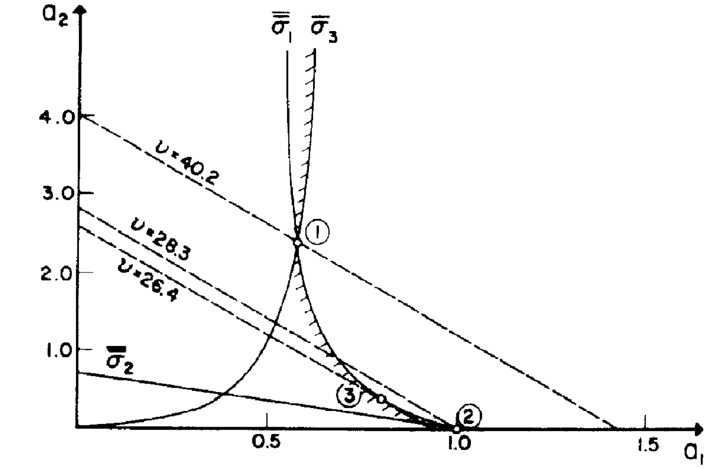
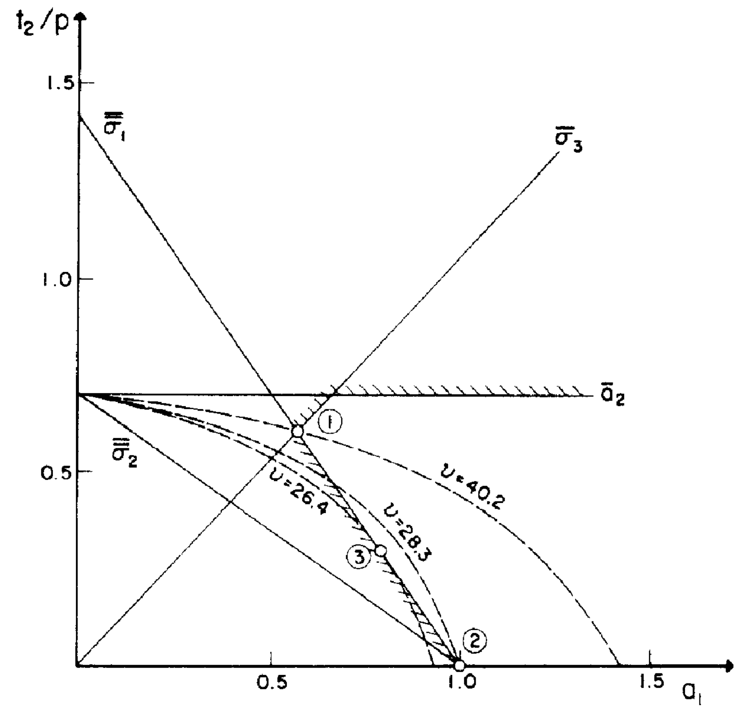


Fig. 1. Symmetric three-bar truss problem.



(a)



(b)

Fig. 2(a). Implicit design space. (b) Explicit design space.

It is this equation which is at the base of the explicit optimization problem. The axial forces in the basic members are with eqn (12)

$$t_1 = 1 - t_2/2c$$

$$t_3 = -t_2/2c. \tag{28}$$

The stresses in the basic structure are obtained by dividing eqns (28) by a_1 and the stress in the redundant bar is computed using eqn (27). This gives the explicit design problem:

$$\text{minimize} \quad v = a_1 l[2/c + ct_2/(c - t_2)] \tag{29a}$$

subject to stress constraints

$$a_1 + ct_2 \geq 1 \tag{29b}$$

$$a_1 + 2ct_2 \geq 1 \tag{29c}$$

$$3a_1 - 4ct_2 \geq 0 \tag{29d}$$

and to side constraints on the design variables

$$a_1 \geq 0 \tag{29e}$$

$$t_2 \leq 0. \tag{29f}$$

The explicit design space is shown in Fig. 2(b). The constraints on $\sigma_i (i = 1, 2, 3)$ and on the cross-section $a_j (j = 1, 2)$ are given by full lines and 3 isovolume contours corresponding to the ones in Fig. 2(a) are shown by dotted-lines. The optimum solution is $a_1 = 0.79$, $t_2 = 0.30$ yielding the minimum volume $v = 26.4$.

6. NUMERICAL EXAMPLE

The 10-bar cantilever truss was originally introduced in Ref. [10] as a test case for numerical experimentation of structural design methods and has since then been used by many authors for that purpose. The structure is submitted to a single loading condition as shown in Fig. 3 and it is required to minimize the volume of the truss subject to stress constraints $-\bar{\sigma}_i = \bar{\bar{\sigma}}_i = 25{,}000$ psi and to lower gage constraints $a_i = 0.1$ in^2 for all the bars of the structure.

The constrained objective function was transformed into an internal penalty function and sequentially minimized using a univariate search technique: conjugate directions in conjunction with parabolic interpolation[11]. The program was run on a CDC-CYBER 730 computer.

The graph in Fig. 4 shows the variation of the volume of the structure as a function of the number of objective function and constraint evaluations (NFUNC). A full line connects design points resulting from unconstrained minimizations and the dashed line joins corresponding designs when scaled down to the most critical constraint. The problem converged in 4 cycles of unconstrained minimizations starting from an initial design $a_i = 10.0$ in^2 for all the elements of the truss. The mean CPU time for evaluating the objective function and its constraints was 0.00345 sec. One should note two important aspects of the present approach. The smooth convergence of the minimization process seems to indicate that the objective function and the constraints are well behaved hypersurfaces in the explicit design space. However, not less more important is the low CPU time required for evaluating the constraints. These characteristics emphasize, in the author's opinion, the potential benefits of this new approach.

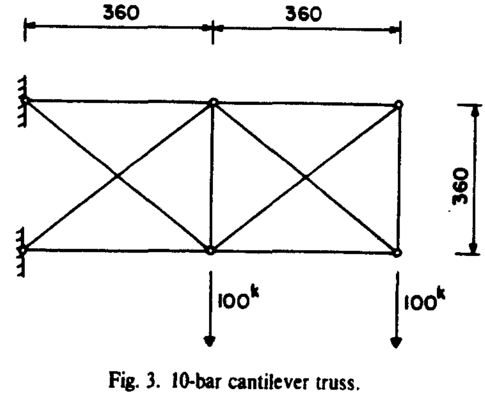


Fig. 3. 10-bar cantilever truss.

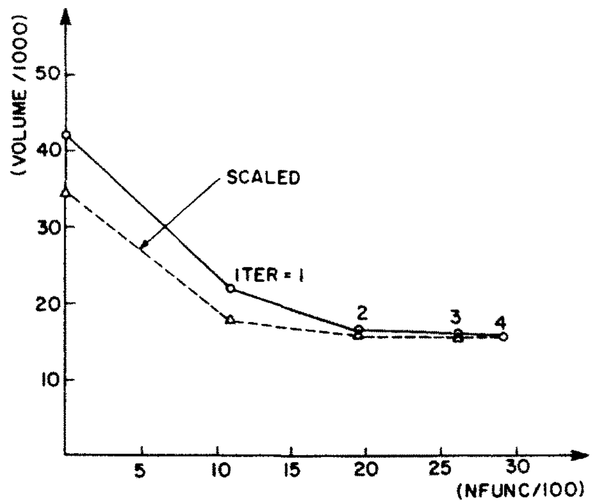


Fig. 4. Convergence of the 10-bar truss problem.

7. CONCLUSIONS

We have shown that an explicit mathematical programming model can be formulated for the design of a truss type structure of given geometry and material properties. This is obtained by expressing the minimization problem in terms of a judicious choice of design variables and behaviour variables. The technique derives roughly from the realization that it is often simpler to match a structure or part of it to imposed force and/or displacement fields than to compute these fields when the structure is fully determined. This gave rise to three explicit analysis methods corresponding to the three types of structural analysis: force method, displacement method and the hybrid (Reissner's) method. Each of these methods lie at the base of an explicit optimization formulation.

The type of minimization technique to be used is at the discretion of the designer and varies with the problem and available computer programs. The important advantage of this new method is that the obstacle of multiple structural reanalysis has been totally eliminated. In fact, there is no place for a separate analysis module in the present scheme.

In the general case of a single loading condition, the technique is, in the author's opinion, superior to any other available structural optimization method. When variables linking is introduced or in the case of multiple loading conditions, the approach can lose some of its attractiveness due to the addition of equality constraints and expansion of the design space. In such cases, its efficiency, when compared to more classical approaches, depends strongly on the problem to be solved.

Acknowledgement—The author would like to thank Mr. Z. Abramowski, graduate student, for the numerical solution of the cantilever truss example.

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